FREE AND SEMI-INERT CELL ATTACHMENTS

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Abstract. Let Y be the space obtained by attaching a finite-type wedge of cells to a simply-connected, finite-type CW-complex.

We introduce the *free* and *semi-inert* conditions on the attaching map which broadly generalize the previously studied *inert* condition. Under these conditions we determine $H_*(\Omega Y;R)$ as an R-module and as an R-algebra respectively. Under a further condition we show that $H_*(\Omega Y;R)$ is generated by Hurewicz images.

As an example we study an infinite family of spaces constructed using only semi-inert cell attachments.

1. Introduction

In this article we will work in the usual category of pointed, simply-connected topological spaces with the homotopy-type of finite-type CW-complexes. We will assume that the ground ring R is either $\mathbb{F}_p$ with $p > 3$ or is a subring of $\mathbb{Q}$ which contains $\frac{1}{6}$.

We are interested in the following problem, perhaps first studied by J.H.C. Whitehead around 1940 [Whi41, Whi39].

The cell attachment problem: *Given a topological space X, what is the effect on the loop space homology and the homotopy-type if one attaches one or more cells to X?*

We approach this problem from the point of view that one is interested in understanding finite cell complexes localized away from finitely many primes [Ani92].

Given a space X and a map $f : W \to X$ where $W = \vee_{j\in J} S^{n_j}$, the *adjunction space*

$$Y = X \cup_f \big(\bigvee_{j\in J} e^{n_j+1}\big),$$

is a homotopy cofibre of f. Let i denote the inclusion $X \hookrightarrow Y$.

The cell attachment problem has been studied in two special cases. One approach is to place a strong condition on the space X. This was done by Anick [Ani89] who considered the case where X is a wedge of spheres. Another approach is to place a strong condition on the attaching map f. This was done by Lemaire and Halperin [Lem78],[HL87] and Félix and Thomas [FT89] who assumed that f is *inert*.

The attaching map $f : W \to X$ is said to be *inert* over a ring R if the induced map $H_*(\Omega i;R) : H_*(\Omega X;R) \to H_*(\Omega Y;R)$ is a surjection.

In this article we generalize these two approaches, with our development following [Ani89]. We generalize Anick's assumption to the more general condition that

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$H_*(\Omega X; R)$ is R-free and is generated by Hurewicz images. This is trivial in the case where $R = \mathbb{Q}$, and we will give conditions under which this holds for more general coefficient rings (see Corollary 1.7). Furthermore we give two generalizations of the inert condition, one of which is strictly stronger than the other.

We now define these conditions in the case where R is a field. We will use the following notation.

Notation 1.1. Given a space X, let L_X denote the image of the Hurewicz map $h_X : \pi_*(\Omega X) \otimes R \to H_*(\Omega X; R)$. L_X is a graded Lie algebra under the commutator bracket of the Pontrjagin product. Given a map $W \to X$, let L_X^W denote the image of the induced Lie algebra map $L_W \to L_X$. Note that the map is omitted from the notation. Let $[L_X^W] \subset L_X$ denote the Lie ideal generated by L_X^W.

Definition 1.2. Define a cell attachment $f : W \to X$ to be *free* if $[L_X^W]$ is a free Lie algebra.

There are examples of spaces given by non-free cell attachments that can be constructed by free cell attachments if one changes the order in which the cells are attached (eg. $\mathbb{CP}^2 \cup_f e^3$ where f is the inclusion $S^2 \hookrightarrow \mathbb{CP}^2$, see [HL96, Example 4.5]). When R is a field, it is conceivable that any cell complex can be constructed using only free cell attachments if one chooses an appropriate cellular structure.

Free cell attachments are convenient to work with because of the following fact about universal enveloping algebras, which we prove as Lemma 3.8. Since it may be of independent interest we state it here. U denotes the universal enveloping algebra functor.

Proposition 1.3. *Let L be a connected, finite-type Lie algebra over a field. Let J be a Lie ideal of L which is a free Lie algebra, $\mathbb{L}W$. Take I to be the two-sided ideal of UL generated by J. Then the multiplication maps $UL \otimes W \to I$ and $W \otimes UL \to I$ are isomorphisms of left and right UL-modules respectively.*

Assume that Y is obtained by a free cell attachment $\bigvee_{j \in J} S^{n_j} \xrightarrow{\bigvee \alpha_j} X$. Let $\hat{\alpha_j}$ denote the adjoint of α_j. We will show that $H_*(\Omega Y; R)$ can be determined by calculating the homology of the following simple differential graded Lie algebra (dgL)

$$\underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle y_j \rangle_{j \in J}, d), \text{ where } dy_j = h_X(\hat{\alpha_j}).$$

In addition to the usual grading, $\underline{\mathbf{L}}$ has a second grading given by letting L_X be in degree 0 and letting each y_j be in degree 1. Remarkably, we will show that for free cell attachments one only needs to calculate $H\underline{\mathbf{L}}$ in degrees 0 and 1.

Let $(H\underline{\mathbf{L}})_i$ denote the component of $H\underline{\mathbf{L}}$ in degree i. For degree reasons, $(H\underline{\mathbf{L}})_0$ acts on $(H\underline{\mathbf{L}})_1$ by the adjoint action.

Definition 1.4. Define f to be a *semi-inert* cell attachment if it is a free cell attachment and $(H\underline{\mathbf{L}})_1$ is a free $(H\underline{\mathbf{L}})_0$-module.

We will see in Section 2 that there is an obvious filtration on $H_*(\Omega Y; R)$. Let $\mathrm{gr}_*(H_*(\Omega Y; R))$ be the *associated graded object*. We will show (see Definition 4.2) that a free cell attachment is semi-inert if and only if $\mathrm{gr}_1(H_*(\Omega Y; R))$ is a free $\mathrm{gr}_0(H_*(\Omega Y; R))$-bimodule.

Theorem A. *Let $Y = X \cup_f \left(\bigvee_{j\in J} e^{n_j+1}\right)$. Assume that f is* free.
(i) Then as algebras

$$\mathrm{gr}(H_*(\Omega Y; R)) \cong U((H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}((H\underline{\mathbf{L}})_1))$$

with $(H\underline{\mathbf{L}})_0 \cong L_X/[L_X^W]$ as Lie algebras.
(ii) Furthermore if f is semi-inert *then for some K',*

$$H_*(\Omega Y; R) \cong U(L_Y^X \amalg \mathbb{L}K')$$

as algebras.

The above result is given more precisely in Theorem 4.3. This theorem is a nearly direct translation of a purely algebraic result given in Theorem 3.12. This algebraic result may have other applications such as the calculation of the mod p Bockstein spectral sequence (BSS) of finite CW-complexes. Scott [Sco02] has shown that for sufficiently large p each term in the mod p BSS of such spaces is a universal enveloping algebra of a dgL.

Corollary 1.5. *Free cell attachments are* nice *in the sense of Hess and Lemaire* [HL96].

When $R = \mathbb{Q}$, Milnor and Moore [MM65] proved the fabulous result that the canonical algebra map $U(\pi_*(\Omega Y) \otimes \mathbb{Q}) \to H_*(\Omega Y; \mathbb{Q})$ is an isomorphism. Scott [Sco03] generalized this result to $R \subset \mathbb{Q}$ for finite CW-complexes when certain primes are invertible in R. For an R-module M, let $FM = M/\operatorname{Torsion}(M)$. Let P denote the primitive elements of $FH_*(\Omega Y; R)$ which are a Lie subalgebra. Scott showed that $UP \xrightarrow{\cong} FH_*(\Omega Y; R)$ and that $F(\pi_*(\Omega Y) \otimes R)$ injects into P. However he showed that in general this injection is not a surjection.

We will give sufficient conditions under which one obtains the desired isomorphism $UF(\pi_*(\Omega Y) \otimes R) \to FH_*(\Omega Y; R)$

In Section 5 we will assume that the Hurewicz map h_X has a right inverse. Using this map we will define the set of *implicit primes* of Y. Intuitively, they are the primes p for which p-torsion is used in the attaching map f. Let

$$\mathcal{S} = \{S^{2m-1}, \Omega S^{2m+1} \mid m \geq 1\}.$$

Let $\prod \mathcal{S}$ be the collection of spaces homotopy equivalent to a weak product of spaces in $\mathcal{S}$.

Theorem B. *Let $Y = X \cup_f \left(\bigvee e^{n_j+1}\right)$. Assume that f is free, that the Hurewicz map h_X has a right inverse, and that the implicit primes are invertible. Then the canonical algebra map*

$$UL_Y \to H_*(\Omega Y; R) \tag{1.1}$$

is a surjection. Furthermore if $R \subset \mathbb{Q}$ then (1.1) *is an isomorphism and localized at R, $\Omega Y \in \prod \mathcal{S}$. If in addition f is semi-inert, then*

$$L_Y \cong H\underline{\mathbf{L}} \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}((H\underline{\mathbf{L}})_1)$$

as Lie algebras, and h_Y has a right inverse.

Note that the surjection of (1.1) implies that $H_*(\Omega Y; R)$ is generated as an algebra by Hurewicz images. Again, more details are given in Theorem 5.5.

Corollary 1.6. *If $R \subset \mathbb{Q}$ then the canonical algebra map*

$$UF(\pi_*(\Omega Y) \otimes R) \to H_*(\Omega Y; R)$$

is an isomorphism.

Corollary 1.7. *If Z is a finite cell complex constructed using only semi-inert cell attachments then localized away from a finite set of primes, $\Omega Z \in \prod \mathcal{S}$.*

It is a long-standing conjecture of Avramov [Avr82] and Félix [FHT84] that if Z has finite LS category then L_Z is either finite dimensional or contains a free Lie subalgebra on two generators. Our final corollary provides further support for this conjecture.

Corollary 1.8. *If $R \subset \mathbb{Q}$, f is a free cell attachment and* $\dim(H\underline{\mathbf{L}})_1 > 1$ *then L_Y contains a free Lie subalgebra on two generators.*

We conclude by giving examples of spaces constructed out of semi-inert cell attachments, together with their Hurewicz images. In particular, we give an infinite family of finite CW-complexes and an uncountable family of finite-type CW-complexes.

Outline of the paper: In Section 2 we will translate the cell attachment problem to a purely algebraic problem using Adams-Hilton models. We then prove our main algebraic results in Section 3. In Section 4 we translate our algebraic results to obtain Theorem A. In Section 5 we prove results about Hurewicz images and homotopy-type to obtain Theorem B. Finally in Section 6 we apply our results to study some examples.

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2. Adams-Hilton models

Let $R = \mathbb{F}_p$ with $p > 3$ or let R be a subring of $\mathbb{Q}$ containing $\frac{1}{6}$.

A simply-connected space X has an Adams-Hilton model [AH56] which we denote $\mathbf{A}(X)$. $\mathbf{A}(X)$ is a connected differential graded algebra (dga) which comes with a chain map $\mathbf{A}(X) \to C_*(\Omega X; R)$ which induces an isomorphism of algebras $H(\mathbf{A}(X)) \xrightarrow{\cong} H_*(\Omega X; R)$.

Given a cell attachment $f : W \to X$ where $W = \bigvee_{j \in J} S^{n_j}$ and $f = \vee_{j\in J} \alpha_j$, let Y be the adjunction space $Y = X \cup_f \left(\bigvee_{j \in J} e^{n_j+1} \right)$. It is a property of Adams-Hilton models that one can choose the following Adams-Hilton model for Y:

$$\mathbf{A}(Y) = \mathbf{A}(X) \amalg \mathbb{T}\langle y_j \rangle_{j \in J},$$

where $\mathbb{T}$ denotes the tensor algebra. The differential on y_j is determined using the attaching map α_j.

Filter $\mathbf{A}(Y)$ by the 'length in y_k's' filtration. That is, let $F_{-1}\mathbf{A}(Y) = 0$, let $F_0\mathbf{A}(Y) = \mathbf{A}(X)$, and for $i \geq 0$, let $F_{i+1}\mathbf{A}(Y) = F_i\mathbf{A}(Y) + \sum_{k=0}^{i} F_k\mathbf{A}(Y) \cdot R\{y_j\}_{j\in J} \cdot F_{i-k}\mathbf{A}(Y)$. This filtration makes $\mathbf{A}(Y)$ a filtered dga.

This filtration induces a first quadrant multiplicative spectral sequence with $E^0_{p,q} = [F_p\mathbf{A}(Y)/F_{p-1}\mathbf{A}(Y)]_{p+q}$ which converges from $\mathrm{gr}(\mathbf{A}(Y))$ to $\mathrm{gr}(H\mathbf{A}(Y))$.

Assume that $H_*(\Omega X; R) \cong UL_X$ as algebras and that it is R-free. Then $(E^1, d^1) \cong (UL_X \amalg U\mathbb{L}\langle y_j \rangle_{j\in J}, d')$, where $\mathbb{L}$ denotes the free Lie algebra and d'

is determined by the induced map $d' : R\{y_j\} \xrightarrow{d} Z\mathbf{A}(X) \twoheadrightarrow H_*(\Omega X; R) \xrightarrow{\cong} UL_X$. It follows from the definition of the differential that $d'y_j = h_X(\hat{\alpha}_j)$ where $\hat{\alpha}_j$ is the adjoint of α_j. Therefore $d'y_j \in L_X$. Thus

$$(E^1, d^1) \cong U\underline{\mathbf{L}}, \text{ where } \underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle y_j \rangle, d').$$

3. Differential Graded Algebra Extensions

Let $R = \mathbb{Q}$ or $\mathbb{F}_p$ where $p > 3$ or let R be a subring of $\mathbb{Q}$ containing $\frac{1}{6}$. All of our R-modules are graded and will be assumed to have finite type.

If $R \subset \mathbb{Q}$, then let P be the set of invertible primes in R and let

$$\tilde{P} = \{p \in \mathbb{Z} \mid p \text{ is prime and } p \notin P\} \cup \{0\}.$$

Notation 3.1. Let $\mathbb{F}_0$ denote $\mathbb{Q}$. If M is an R-module then for each $p \in \tilde{P}$ we will denote $M \otimes \mathbb{F}_p$ by $\bar{M}$ omitting p from the notation. Similarly if d is a differential we will denote $d \otimes \mathbb{F}_p$ by $\bar{d}$.

Let (A, d) be a connected finite-type differential graded algebra (dga) over R which is R-free. Let ZA denote the subalgebra of cycles of A. Let V_1 be a connected finite-type free R-module and let $d : V_1 \to ZA$. Then there is a canonical dga extension $\mathbf{B} = (A \amalg \mathbb{T}V_1, d)$.

Assume that for some Lie algebra L_0 which is a free R-module, $H(A, d) \cong UL_0$ as algebras. There is an induced map $d' : V_1 \xrightarrow{d} ZA \to HA \xrightarrow{\cong} UL_0$. Assume that L_0 can be chosen such that $d'V_1 \subset L_0$.

Taking $d'L_0 = 0$, there is a canonical differential graded Lie algebra (dgL)

$$\underline{\mathbf{L}} = (L_0 \amalg \mathbb{L}V_1, d').$$

Then $\underline{\mathbf{L}}$ is a bigraded dgL where the usual grading is called dimension and a second grading, called degree, is given by taking L_0 and V_1 to be in degrees 0 and 1 respectively. Then the differential d has bidegree $(-1, -1)$.

Notation 3.2. Subscripts of bigraded objects will denote degree, eg. M_0 is the component of M in degree 0.

The following lemma is a well-known fact, and the subsequent lemma is part of lemmas from [Ani89]. We remind the reader that all of our R-modules have finite type.

Lemma 3.3. *Let $R \subset \mathbb{Q}$. A homomorphism $f : M \to N$ is an isomorphism if and only if for each $p \in \tilde{P}$, $f \otimes \mathbb{F}_p$ is an isomorphism.*

Let L be a connected bigraded dgL. The inclusion $L \hookrightarrow UL$ induces a natural map

$$\psi : UHL \to HUL. \tag{3.1}$$

Lemma 3.4 ([Ani89, Lemmas 4.1 and 4.3])**.** *Let $R = \mathbb{F}_p$ with $p > 3$ or let $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Suppose that HUL is R-free in degrees 0 and 1. Then HL is R-free in degrees 0 and 1 and the map ψ in (3.1) is an isomorphism in degrees 0 and 1.*

$\mathbf{B}$ is a filtered dga under the increasing filtration given by $F_{-1}\mathbf{B} = 0$, $F_0\mathbf{B} = A$, and for $i \geq 0$, $F_{i+1}\mathbf{B} = \sum_{j=0}^{i} F_j\mathbf{B} \cdot V_1 \cdot F_{i-j}\mathbf{B}$. Letting $E^0_{p,q}(\mathbf{B}) = [F_p(\mathbf{B})/F_{p-1}(\mathbf{B})]_{p+q}$ gives a first quadrant spectral sequence of algebras:

$$(E^0(\mathbf{B}), d^0) = \mathrm{gr}(\mathbf{B}) \implies E^\infty = \mathrm{gr}(H\mathbf{B}).$$

It is easy to check that $(E^1, d^1) \cong U\underline{\mathbf{L}}$ and hence $E^2 \cong HU\underline{\mathbf{L}}$. The following theorem follows from the main result of Anick's thesis [Ani82, Theorem 3.7]. Anick's theorem holds under either of two hypotheses. We will use only one of these.

Recall that the Hilbert series of an $\mathbb{F}$-module is given by the power series $A(z) = \Sigma_{n=0}^{\infty}(\mathrm{Rank}_{\mathbb{F}} A_n)z^n$. Assuming that $A_0 \neq 0$, the notation $(A(z))^{-1}$ denotes the power series $1/(A(z))$.

Theorem 3.5. *Let $R = \mathbb{F}$. If the two-sided ideal $(d'V_1) \subset UL_0$ is a free UL_0-module then the above spectral sequence collapses at the E^2 term. That is,* $\mathrm{gr}(H\mathbf{B}) \cong HU\underline{\mathbf{L}}$ *as algebras. Furthermore the multiplication map*

$$\nu : \mathbb{T}(\psi(H\underline{\mathbf{L}})_1) \otimes (HU\underline{\mathbf{L}})_0 \to HU\underline{\mathbf{L}}$$

is an isomorphism and $(HU\underline{\mathbf{L}})_0 \cong UL_0/(d'V_1)$. In addition,

$$H\mathbf{B}(z)^{-1} = HU\underline{\mathbf{L}}(z)^{-1} = (1+z)(HU\underline{\mathbf{L}})_0(z)^{-1} - z(UL_0)(z)^{-1} - V_1(z). \tag{3.2}$$

Proof. [Ani82, Theorem 3.7] shows that the spectral sequence collapses as claimed and that the multiplication map $\mathbb{T}W \otimes (HU\underline{\mathbf{L}})_0 \to HU\underline{\mathbf{L}}$ is an isomorphism where W is a basis for $(HU\underline{\mathbf{L}})_1$ as a right $(HU\underline{\mathbf{L}})_0$-module. By Lemma 3.4 and the Poincaré-Birkhoff-Witt Theorem the homomorphism $\psi(H\underline{\mathbf{L}})_1 \otimes (HU\underline{\mathbf{L}})_0 \to (HU\underline{\mathbf{L}})_1$ induced by multiplication in $HU\underline{\mathbf{L}}$ is an isomorphism. So we can let $W = \psi(H\underline{\mathbf{L}})_1$.

The remainder of the theorem follows directly from [Ani82, Theorem 3.7]. □

Corollary 3.6. *If $R \subset \mathbb{Q}$ and for each $p \in \tilde{P}$, the two-sided ideal $(\bar{d}\bar{V}_1) \subset U\bar{L}_0$ is a free $U\bar{L}_0$-module, then $H\mathbf{B}$ is R-free if and only if $HU\underline{\mathbf{L}}$ is R-free if and only if $L_0/[d'V_1]$ is R-free.*

Proof. First $(HU\underline{\mathbf{L}})_0 \cong UL_0/(d'V_1) \cong U(L_0/[d'V_1])$. So $(HU\underline{\mathbf{L}})_0$ is R-free if and only if $L_0/[d'V_1]$ is R-free. Since UL_0 and V_1 are R-free, the corollary follows from (3.2). □

We now prove a version of Theorem 3.5 for subrings of $\mathbb{Q}$.

Theorem 3.7. *Let $R \subset \mathbb{Q}$. If $L_0/[d'V_1]$ is R-free and for each $p \in \tilde{P}$, the two-sided ideal $(\bar{d}\bar{V}_1) \subset U\bar{L}_0$ is a free $U\bar{L}_0$-module, then $H\mathbf{B}$ is R-free and the multiplication map*

$$\nu : \mathbb{T}(\psi(H\underline{\mathbf{L}})_1) \otimes (HU\underline{\mathbf{L}})_0 \to HU\underline{\mathbf{L}}$$

is an isomorphism. Also $\mathrm{gr}(H\mathbf{B}) \cong HU\underline{\mathbf{L}}$ *as algebras and $(HU\underline{\mathbf{L}})_0 \cong UL_0/(d'V_1)$.*

Proof. Since $L_0/[d'V_1]$ is R-free, by Corollary 3.6 so are $HU\underline{\mathbf{L}}$ and $H\mathbf{B}$. It follows from the Universal Coefficient Theorem that $\forall p \in \tilde{P}$, $H\mathbf{B} \otimes \mathbb{F}_p \cong H(\mathbf{B} \otimes \mathbb{F}_p)$ and $HU\underline{\mathbf{L}} \otimes \mathbb{F}_p \cong HU(\underline{\mathbf{L}} \otimes \mathbb{F}_p)$. In particular $\forall p \in \tilde{P}$, $(HU\underline{\mathbf{L}})_0 \otimes \mathbb{F}_p \cong (HU(\underline{\mathbf{L}} \otimes \mathbb{F}_p))_0$. Using Lemma 3.4,

$$\psi(H\underline{\mathbf{L}})_1 \otimes \mathbb{F}_p \cong (H\underline{\mathbf{L}})_1 \otimes \mathbb{F}_p \cong H(\underline{\mathbf{L}} \otimes \mathbb{F}_p)_1 \cong \psi H(\underline{\mathbf{L}} \otimes \mathbb{F}_p)_1.$$

Thus $\forall p \in \tilde{P}$,

$$\nu \otimes \mathbb{F}_p : \mathbb{T}(\psi(H\underline{\mathbf{L}})_1 \otimes \mathbb{F}_p) \otimes (HU\underline{\mathbf{L}})_0 \otimes \mathbb{F}_p \to H\mathbf{B} \otimes \mathbb{F}_p$$

corresponds under these isomorphisms to the multiplication map

$$\mathbb{T}(\psi(H(\underline{\mathbf{L}} \otimes \mathbb{F}_p))_1) \otimes (HU(\underline{\mathbf{L}} \otimes \mathbb{F}_p))_0 \to H(\mathbf{B} \otimes \mathbb{F}_p).$$

But this is an isomorphism by Theorem 3.5. Therefore ν is an isomorphism by Lemma 3.3.

The last two isomorphisms also follow from Theorem 3.5. □

The next lemma will prove that if the Lie ideal $[d'V_1] \subset L_0$ is a free Lie algebra then the hypothesis in Anick's Theorem (Theorem 3.5) holds. That is, $(d'V_1)$ is a free UL_0-module.

Lemma 3.8. *Given a dgL L over a field $\mathbb{F}$, denote UL by A. Let J be a Lie ideal of L which is a free Lie algebra, $\mathbb{L}W$. Take I to be the two-sided ideal of A generated by J. Then the multiplication maps $A \otimes W \to I$ and $W \otimes A \to I$ are isomorphisms of left and right A-modules respectively.*

Proof. From the short exact sequence of Lie algebras

$$0 \to J \to L \to L/J \to 0$$

we get the short exact of sequence of Hopf algebras

$$\mathbb{F} \to U(J) \to U(L) \to U(L/J) \to \mathbb{F}$$

and so $UL \cong UJ \otimes U(L/J)$ as $\mathbb{F}$-modules. Since J is a free Lie algebra $\mathbb{L}W$, $UJ \cong \mathbb{T}W$. It is also a basic fact that $U(L/J) \cong UL/I$. Hence we have that

$$A \cong TW \otimes A/I \tag{3.3}$$

as $\mathbb{F}$-modules. Furthermore

$$A \cong I \oplus A/I \tag{3.4}$$

as $\mathbb{F}$-modules.

Let $M(z)$ denote the Hilbert series for the $\mathbb{F}$-module M, and to simplify the notation let $B = A/I$. Then from equations (3.3) and (3.4) we have the following (using $(TW)(z) = 1/(1 - W(z))$).

$$B(z) = A(z)(1 - W(z)), \quad I(z) = A(z) - B(z).$$

Combining these we have $I(z) = A(z)W(z)$. That is, $I \cong A \otimes W$ as $\mathbb{F}$-modules.

Let $\mu : A \otimes W \to I$ be the multiplication map. To show that it is an isomorphism it remains to show that it is either injective or surjective.

We claim that μ is surjective. Since I is the ideal in A generated by W, any $x \in I$ can be written as

$$x = \Sigma_i a_i w_i b_{i_1} \cdots b_{i_{m_i}}, \text{ where } a_i \in A,\ w_i \in W \text{ and } b_{i_k} \in L. \tag{3.5}$$

Each such expression gives a sequence of numbers $\{m_i\}$. Let $M(x) = \min\,[\max_i(m_i)]$, where the minimum is taken over all possible ways of writing x as in (3.5). We claim that $M(x) = 0$.

Assume that $M(x) = t > 0$. Then $x = x' + \Sigma_i a_i w_i b_{i_1} \cdots b_{i_t}$, where $M(x') < t$. Now $w_i b_{i_1} = [w_i, b_{i_1}] \pm b_{i_1} w_i$. Furthermore since J is a Lie ideal $[w_i, b_{i_1}] \in J \cong \mathbb{L}W$, so

$$[w_i, b_{i_1}] = \Sigma_j c_j [[w_{j_1}, \ldots, w_{j_{n_j}}] = \Sigma_k d_k w_{k_1} \cdots w_{k_{N_k}} = \Sigma_l a_l w_l,$$

where $a_l \in A$ and $w_l \in W$. So $x = x' + \Sigma_i \Sigma_{l_i} a_i a_{l_i} w_{l_i} b_{i_2} \cdots b_{i_t}$. But this is of the form in (3.5) and shows that $M(x) < t$ which is a contradiction.

Therefore for each $x \in I$, $M(x) = 0$ and we can write $x = \Sigma_i a_i w_i$ where $a_i \in A$ and $w_i \in W$. Then $x \in \mathrm{im}(\mu)$ and hence μ is an isomorphism.

Since A is associative, μ is a map of left A-modules.

The second isomorphism follows similarly. $\square$

We are now almost ready to prove our main algebraic results. Recall that $\mathbf{B} = (A \amalg \mathbb{T}V_1, d)$ where $dA \subset A$ and $dV_1 \subset ZA$. Also $H(A,d) \cong UL_0$ as algebras and if $d' : V_1 \to UL_0$ is the induced map then $d'(V_1) \subset L_0$. Let $\underline{\mathbf{L}} = (L_0 \amalg \mathbb{L}V_1, d')$ with $d'L_0 = 0$.

We introduce the following terminology.

Definition 3.9. If R is a field say $\mathbf{B}$ is *free* dga extension if the Lie ideal $[d'V_1] \subset L_0$ is a free Lie algebra. If $R \subset \mathbb{Q}$ say $\mathbf{B}$ is *free* dga extension if $L_0/[d'V_1]$ is R-free and, using Notation 3.1, for every $p \in \tilde{P}$, the Lie ideal $[\bar{d}'\bar{V}_1] \subset \bar{L}_0$ is a free Lie algebra.

Definition 3.10. In either of the cases of the previous definition we say $\mathbf{B}$ is a *semi-inert* dga extension if in addition there is a free R-module K such that $(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K$. At the end of this section we will give two simpler equivalent conditions (see Lemma 3.13).

Note that it follows from [HL87, Theorem 3.3] and [FT89, Theorem 1] that the semi-inert condition is a generalization of the inert condition.

Recall that there is a map $\underline{\psi} : UH\underline{\mathbf{L}} \to HU\underline{\mathbf{L}}$. $\mathbf{B}$ is a filtered dga under the increasing filtration given by $F_{-1}\mathbf{B} = 0$, $F_0\mathbf{B} = A$, and for $i \geq 0$, $F_{i+1} = \sum_{k=0}^{i} F_k\mathbf{B} \cdot V_1 \cdot F_{i-k}\mathbf{B}$. There is an induced filtration on $H\mathbf{B}$. We prove one last lemma.

Lemma 3.11. *There exists a quotient map*

$$f : F_1H\mathbf{B} \to (HU\underline{\mathbf{L}})_1. \tag{3.6}$$

Given $\bar{w} \in (H\underline{\mathbf{L}})_1$ there exists a cycle $w \in F_1\mathbf{B}$ such that $f([w]) = \bar{w}$.

Proof. By Theorem 3.5 or Theorem 3.7, $(\mathrm{gr}(H\mathbf{B}))_1 \cong (HU\underline{\mathbf{L}})_1$. So there is a quotient map

$$f : F_1H\mathbf{B} \twoheadrightarrow (\mathrm{gr}(H\mathbf{B}))_1 \xrightarrow{\cong} (HU\underline{\mathbf{L}})_1.$$

By Lemma 3.4 $(H\underline{\mathbf{L}})_1 \cong (\underline{\psi}H\underline{\mathbf{L}})_1 \subset (HU\underline{\mathbf{L}})_1$. So for $\bar{w}$ one can choose a representative cycle $w \in ZF_1\mathbf{B}$ such that $f([w]) = \bar{w}$. □

Recall that $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Also all of our R-modules are connected, R-free, and have finite type. Let (A,d) be a dga and let V_1 be a R-module with a map $d : V_1 \to A$. Assume that there exists a Lie algebra L_0 such that $H(A,d) \cong UL_0$ as algebras and $d'V_1 \subset L_0$ where d' is the induced map.

Theorem 3.12. *Let $\mathbf{B} = (A \amalg \mathbb{T}V_1, d)$. Assume that $\mathbf{B}$ is a* free *dga extension in the sense of Definition 3.9. Let $\underline{\mathbf{L}} = (L_0 \amalg \mathbb{L}V_1, d')$.*
(a) Then as algebras

$$\mathrm{gr}(H\mathbf{B}) \cong U((H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1)$$

with $(H\underline{\mathbf{L}})_0 \cong L_0/[d'V_1]$ as Lie algebras. If $R \subset \mathbb{Q}$ then additionally $(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong \underline{\psi}H\underline{\mathbf{L}}$ as Lie algebras.
(b) Furthermore if $\mathbf{B}$ is semi-inert *(that is, there is an R-module K such that $(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K$) then as algebras*

$$H\mathbf{B} \cong U((H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K')$$

for some $K' \subset F_1H\mathbf{B}$ such that $f : K' \xrightarrow{\cong} K$, where f is the quotient map in Lemma 3.11.

Proof. (a) If $R = \mathbb{F}_p$ then by Lemma 3.8, $(d'V_1) \subset UL_0$ is a free UL_0-module. If $R \subset \mathbb{Q}$ then by Lemma 3.8, for each $p \in \tilde{P}$, $(\bar{d}'\bar{V}_1) \subset U\bar{L}_0$ is a free $U\bar{L}_0$-module. So we can apply either Theorem 3.5 or Theorem 3.7 to show that $\mathrm{gr}(H\mathbf{B}) \cong HU\underline{\mathbf{L}}$ as algebras and that the multiplication map

$$\nu : \mathbb{T}(\underline{\psi}(H\underline{\mathbf{L}})_1) \otimes (HU\underline{\mathbf{L}})_0 \to HU\underline{\mathbf{L}}$$

is an isomorphism.

By Lemma 3.4 $(HU\underline{\mathbf{L}})_0 \cong U(H\underline{\mathbf{L}})_0$ and $\underline{\psi}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_1$. By the definition of homology $(H\underline{\mathbf{L}})_0 \cong L_0/[d'V_1]$.

Let $N = \underline{\psi}(H\underline{\mathbf{L}})$. Then N_0 acts on N_1 so we can define

$$L' = N_0 \ltimes \mathbb{L}N_1.$$

Note that $L'_0 = N_0$ and $L'_1 = N_1$. There is a Lie algebra map $u : L' \to N$ and an induced algebra map $\tilde{u} : UL' \xrightarrow{Uu} UN \to HU\underline{\mathbf{L}}$.

Recall that as R-modules, $L' \cong N_0 \times \mathbb{L}N_1$. The Poincaré-Birkhoff-Witt Theorem shows that the multiplication map

$$\phi : \mathbb{T}N_1 \otimes (HU\underline{\mathbf{L}})_0 \xrightarrow{\cong} U\mathbb{L}N_1 \otimes UN_0 \to UL'$$

is an isomorphism. Since $\tilde{u}$ is an algebra map, $\nu = \tilde{u} \circ \phi$. Thus $\tilde{u}$ is an isomorphism. Therefore $HU\underline{\mathbf{L}} \cong UL'$ as algebras and hence $\mathrm{gr}(H\mathbf{B}) \cong UL'$. If $R = \mathbb{F}_p$ then this finishes (a).

If $R \subset \mathbb{Q}$ then we will show that $u : L' \to N$ is an isomorphism. Let $\iota : N \hookrightarrow HU\underline{\mathbf{L}}$ be the inclusion. Since the composition $L' \xrightarrow{u} N \xrightarrow{\iota} HU\underline{\mathbf{L}} \xrightarrow{\cong} UL'$ is the canonical inclusion $L' \hookrightarrow UL'$, u is injective. The inclusion $L' \to UL'$ splits as R-modules; so as R-modules $N \cong L' \oplus N/L'$. Since L' and N are R-free, so is N/L'.

Recall that the composition $\iota \circ u$ induces the isomorphism $\tilde{u} : UL' \xrightarrow{Uu} UN \to HU\underline{\mathbf{L}}$. Tensor these maps with $\mathbb{Q}$ to get the commutative diagram

$$\begin{tikzcd} UL' \otimes \mathbb{Q} \arrow[r, "Uu\otimes\mathbb{Q}"] \arrow[dr, "\cong"'] & UN \otimes \mathbb{Q} \arrow[d] \\ & HU\underline{\mathbf{L}} \otimes \mathbb{Q} \end{tikzcd} \quad . \tag{3.7}$$

It is a classical result that the natural map

$$\underline{\psi}_{\mathbb{Q}} : UH(\underline{\mathbf{L}} \otimes \mathbb{Q}) \xrightarrow{\cong} HU(\underline{\mathbf{L}} \otimes \mathbb{Q}) \tag{3.8}$$

is an isomorphism. Notice that

$$N \otimes \mathbb{Q} = (\underline{\psi}H\underline{\mathbf{L}}) \otimes \mathbb{Q} \cong \underline{\psi}_{\mathbb{Q}}H(\underline{\mathbf{L}} \otimes \mathbb{Q}) \cong H(\underline{\mathbf{L}} \otimes \mathbb{Q})$$

and $HU\underline{\mathbf{L}} \otimes \mathbb{Q} \cong HU(\underline{\mathbf{L}} \otimes \mathbb{Q})$. Under these isomorphisms the vertical map in (3.7) corresponds to the isomorphism in (3.8).

Therefore $Uu \otimes \mathbb{Q}$ is an isomorphism and hence $u \otimes \mathbb{Q}$ is surjective. As a result $\operatorname{coker} u = N/L'$ is a torsion R-module. But we have already shown that N/L' is R-free. Thus $N/L' = 0$ and $N \cong L'$. Hence $HU\underline{\mathbf{L}} \cong UN$.

(b) Recall that N_0 acts on $N_1 = (\underline{\psi}H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_1$ via the adjoint action. Assume that $\mathbf{B}$ is semi-inert. That is, there exists $\{\bar{w}_i\} \subset N_1$ such that $L' \cong N_0 \amalg \mathbb{L}K$,

where $K = R\{\bar{w}_i\}$. Recall from (a) that $HU\underline{\mathbf{L}} \cong \mathrm{gr}(H\mathbf{B})$ and that the inclusions $N_0 \subset HU\underline{\mathbf{L}}$ and $\bar{w}_i \in HU\underline{\mathbf{L}}$ induce a Lie algebra map

$$u : L' \to \mathrm{gr}(H\mathbf{B}).$$

By Lemma 3.11, there exists $w_i \in F_1\mathbf{B}$ such that $f([w_i]) = \bar{w}_i$ where f is the map in (3.6). Let $K' = R\{[w_i]\} \subset F_1H\mathbf{B}$, and let $L'' = N_0 \amalg \mathbb{L}K'$. Then $f : K' \xrightarrow{\cong} K$ and f induces an isomorphism $L'' \xrightarrow{\cong} L'$.

By part (a), $N_0 \subset (\mathrm{gr}\, H\mathbf{B})_0$. Since $F_{-1}H\mathbf{B} = 0$, $(\mathrm{gr}\, H\mathbf{B})_0 \hookrightarrow H\mathbf{B}$, so $N_0 \hookrightarrow H\mathbf{B}$. Since N_0, $K' \hookrightarrow H\mathbf{B}$, there are induced maps

$$\begin{tikzcd} L'' \arrow[r, "\eta"] \arrow[d] & H\mathbf{B} \\ UL'' \arrow[ur, "\theta"'] & \end{tikzcd},$$

where η is a Lie algebra map and θ is an algebra map.

Grade L'' by letting N_0 be in degree 0 and K' be in degree 1. This also filters L''. Then η is a map of filtered objects.

From this we get the following commutative diagram

$$\begin{tikzcd} \mathrm{gr}(L'') \arrow[r, "\mathrm{gr}(\eta)"] \arrow[d] & \mathrm{gr}(H\mathbf{B}) \\ \mathrm{gr}(UL'') \arrow[ur, "\mathrm{gr}(\theta)"] \arrow[d, "\cong"'] & \\ U\,\mathrm{gr}(L'') \arrow[uur, "\rho"'] & \end{tikzcd}.$$

Now $\mathrm{gr}(L'') \cong L'' \cong L'$ and $\mathrm{gr}(\eta)$ corresponds to u under this isomorphism. So ρ corresponds to $\tilde{u}$ which is an isomorphism. Thus $\mathrm{gr}(\theta)$ is an isomorphism, and hence θ is an isomorphism. Therefore $H\mathbf{B} \cong UL''$ which finishes the proof. □

As promised we now give two simpler equivalent conditions for semi-inertness.

Lemma 3.13. *Let $\mathbf{B}$ be a free dga extension (in the sense of Definition 3.9). Then the following conditions are equivalent:*
(a) $(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K$ for some free R-module $K \subset (H\underline{\mathbf{L}})_1$,
(b) $(H\underline{\mathbf{L}})_1$ is a free $(H\underline{\mathbf{L}})_0$-module, and
(c) $\mathrm{gr}_1(H\mathbf{B})$ is a free $\mathrm{gr}_0(H\mathbf{B})$-bimodule.

Proof. (b) $\implies$ (a) Let K be a basis for $(H\underline{\mathbf{L}})_1$ as a free $(H\underline{\mathbf{L}})_0$-module. Then $(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K$.
(a) $\implies$ (c) Since $\mathbf{B}$ is a free dga extension, by Theorem 3.12(a), $\mathrm{gr}_*(H\mathbf{B}) \cong U\left((H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1\right)$. So by (a),

$$\mathrm{gr}_*(H\mathbf{B}) \cong U\left((H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K\right) \cong \mathrm{gr}_0(H\mathbf{B}) \amalg \mathbb{T}K,$$

for some free R-module $K \subset (H\underline{\mathbf{L}})_1$. Therefore

$$\mathrm{gr}_1(H\mathbf{B}) \cong \left[\mathrm{gr}_0(H\mathbf{B}) \amalg \mathbb{T}K\right]_1 \cong \mathrm{gr}_0(H\mathbf{B}) \otimes K \otimes \mathrm{gr}_0(H\mathbf{B}).$$

(c) $\implies$ (b) Let $L' = (H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1$. Then by Theorem 3.12(a), $\mathrm{gr}_*(H\mathbf{B}) \cong UL'$ and $\mathrm{gr}_1(H\mathbf{B}) \cong (UL')_1$. By (c), $(UL')_1$ is a free $(UL')_0$-bimodule. Then we claim

that it follows that N_1 is a free N_0-module. Indeed, if there is a nontrivial degree one relation in L' then there is a corresponding nontrivial degree one relation in UL'. $\square$

4. Cell-attachments

Let $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Let X be a finite-type simply-connected CW-complex such that $H_*(\Omega X; R)$ is torsion-free and as algebras $H_*(\Omega X; R) \cong UL_X$ where L_X is the Lie algebra of Hurewicz images. Let $W = \bigvee_{j\in J} S^{n_j}$ be a finite-type wedge of spheres and let $f : W \to X$. Let $Y = X \cup_f \left(\bigvee_{j\in J} e^{n_j+1}\right)$. Using the Adams-Hilton models of Section 2, we defined a Lie algebra $\underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle y_j\rangle, d')$. Mirroring Definitions 3.9 and 3.10, we introduce the following terminology.

Definition 4.1. If R is a field call f a *free* cell attachment if the Lie ideal $[L_X^W] \subset L_X$ is a free Lie algebra. If $R \subset \mathbb{Q}$ call f a *free* cell attachment if $L_X/[L_X^W]$ is R-free and for every $p \in \tilde{P}$, the Lie ideal $[\bar{L}_X^W] \subset \bar{L}_X$ is a free Lie algebra.

Definition 4.2. In either of the cases of the previous definition say that f is a *semi-inert* cell attachment if in addition one of the following three equivalent (by Lemma 3.13] conditions is satisfied:
(a) $\mathrm{gr}_1(H\mathbf{A}(Y))$ is a free $\mathrm{gr}_0(H\mathbf{A}(Y))$-bimodule,
(b) $(H\underline{\mathbf{L}})_1$ is a free $(H\underline{\mathbf{L}})_0$-module, or
(c) there is a free R-module K such that

$$(H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1 \cong (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}K.$$

Theorem 3.12 gives most of the following topological result directly.

Theorem 4.3. *Let $Y = X \cup_f \left(\bigvee_{j\in J} e^{n_j+1}\right)$ and let $\underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle y_j\rangle, d')$. Assume that f is* free*.*
(a) Then $H_*(\Omega Y; R)$ and $\mathrm{gr}(H_*(\Omega Y; R))$ are R-free and as algebras

$$\mathrm{gr}(H_*(\Omega Y; R)) \cong U(L_Y^X \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1)$$

with $L_Y^X \cong L_X/[L_X^W]$ as Lie algebras.
(b) Furthermore if f is semi-inert *then as algebras*

$$H_*(\Omega Y; R) \cong U(L_Y^X \amalg \mathbb{L}K')$$

for some $K' \subset F_1H_*(\Omega Y; R)$.

Proof. It remains to show that $(H\underline{\mathbf{L}})_0 \cong L_Y^X$. By Theorem 3.12 we have the algebra isomorphism

$$\mathrm{gr}(H\mathbf{A}(Y)) \cong U((H\underline{\mathbf{L}})_0 \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1)$$

with $(H\underline{\mathbf{L}})_0 \cong L_X/[L_X^W]$. Therefore

$$F_0H\mathbf{A}(Y) \cong (\mathrm{gr}(H\mathbf{A}(Y)))_0 \cong U(H\underline{\mathbf{L}})_0 \cong U(L_X/[L_X^W]). \tag{4.1}$$

The inclusion $i : \mathbf{A}(X) \xrightarrow{\cong} F_0\mathbf{A}(Y)$ induces a map $H(i) : H\mathbf{A}(X) \to F_0H\mathbf{A}(Y)$. Now under the isomorphism (4.1) and $UL_X \xrightarrow{\cong} H\mathbf{A}(X)$ the map $H(i)$ corresponds to a map $UL_X \to U(L_X/[L_X^W])$ where $U(L_X/[L_X^W]) \subset UL_Y$. It is easy to check that this is the canonical map. In other words $L_Y^X \cong L_X/[L_X^W]$. Therefore $(H\underline{\mathbf{L}})_0 \cong L_Y^X$. $\square$

Corollary 1.5 follows from Theorem 4.3.

Proof of Corollary 1.5. The cell attachment f is nice in the sense of Hess and Lemaire [HL96] if and only if $UL_X/(L_X^W)$ injects in $H_*(\Omega Y;R)$. Recall the standard fact that $UL_X/(L_X^W) \cong U(L_X/[L_X^W])$. By Theorem 4.3,

$$U(L_X/[L_x^W]) \cong UL_Y^X \cong \mathrm{gr}_0(H_*(\Omega Y;R))$$

which injects in $H_*(\Omega Y;R)$. $\square$

5. Hurewicz images

Let $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ be a subring containing $\frac{1}{6}$.

Recall that we have a homotopy cofibration $W \xrightarrow{f} X \to Y$ where $W = \bigvee_{j\in J} S^{m_j}$ is a finite-type wedge of spheres, $f = \bigvee_{j\in J} \alpha_j$, $H_*(\Omega X;R)$ is torsion-free, and as algebras $H_*(\Omega X;R) \cong UL_X$. Let $\hat{\alpha}$ denote the adjoint of α.

Assume that f is free. That is, the Lie ideal $[L_X^W]$ is a free Lie algebra.

Recall that $h_X : \pi_*(\Omega X)\otimes R \to L_X \subset H_*(\Omega X;R)$ is a Lie algebra map. In order to identify Hurewicz images in $H_*(\Omega Y;R)$ we will need to be able to construct maps from information about the loop space homology. In particular, we will need to assume that there exists a Lie algebra map $\sigma_X : L_X \to \pi_*(\Omega X) \otimes R$ such that $h_X \circ \sigma_X = id_{L_X}$. This map exists if $R = \mathbb{Q}$ or if X is a wedge of spheres. We will give a sufficient condition for the existence of this map later in this section.

If R is a subring of $\mathbb{Q}$ with invertible primes $P \supset \{2,3\}$, then we may need to exclude those primes p for which an attaching map $\alpha_j \in \pi_*(X)$ includes a term with p-torsion. Following Anick [Ani89] we define the set of *implicit primes* below.

Definition 5.1. By the Milnor-Moore theorem [MM65], h_X, σ_X are rational isomorphisms, so $\mathrm{im}(\sigma_X \circ h_X - id)$ is a torsion element of $\pi_*(\Omega X)\otimes R$. Let $\gamma_j = \sigma_X h_X(\hat{\alpha_j}) - \hat{\alpha}_j$ where $\hat{\alpha}_j : S^{n_j-1} \to \Omega X$ is the adjoint of α_j. Then $t_j\gamma_j = 0 \in \pi_*(\Omega X)\otimes R$ for some $t_j > 0$. Let t_j be the smallest such integer. Define P_Y, the set of *implicit primes* of Y as follows. A prime p is in P_Y if and only if $p \in P$ or $p|t_j$ for some $j \in J$.

One can verify that the implicit primes have the following properties.

Lemma 5.2. *(a) Let $\{x_i\}$ be a set of Lie algebra generators for L_X and let $\beta_i = \sigma_X x_i$. If all of the attaching maps are R-linear combinations of iterated Whitehead products of the maps β_i, then $P_Y = P$.*
(b) If $P = \{2,3\}$ and $n = \dim(Y)$ then the implicit primes are bounded by $\max(3, n/2)$.

By replacing R with $R' = \mathbb{Z}[{P_Y}^{-1}]$ if necessary, we may assume that the implicit primes are invertible. This implies that for all $j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$, and hence $\sigma_X(dy_j) = \hat{\alpha_j}$.

Remark 5.3. If $R = \mathbb{F}_p$ then we will also need that for all $j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$. So that we can state the cases $R = \mathbb{F}_p$ and $R \subset \mathbb{Q}$ together, when $R = \mathbb{F}_p$ and we say *the implicit primes are invertible* we mean that for all $j \in J$, $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$.

We now consider both cases $R = \mathbb{F}_p$ or $R \subset \mathbb{Q}$. Recall that in Section 2 we defined the differential graded Lie algebra

$$\underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle y_j\rangle_{j\in J}, d').$$

By Theorem 4.3, $H_*(\Omega Y;R)$ is torsion-free and $\mathrm{gr}(H_*(\Omega Y;R)) \cong U(L_Y^X \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1)$ as algebras. From this we want to show that $H_*(\Omega Y;R) \cong UL_Y$ as algebras.

This situation closely resembles that of torsion-free spherical two-cones, and we will generalize Anick's proof for that situation [Ani89].

The proof of the following is the same as the proof of [Ani89, Claim 4.7], so we will only sketch it here. See [Bub03, Chapter 7] for more details.

Proposition 5.4. *Let $W \to X \to Y$ and $\underline{\mathbf{L}}$ be as above with $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$, $\forall j$. Then there exists an injection of L_Y^X-modules*

$$u_1 : (H\underline{\mathbf{L}})_1 \hookrightarrow (\mathrm{gr}(L_Y))_1.$$

Proof sketch. Let $\bar{w}_i \in (H\underline{\mathbf{L}})_1$ be a homology class in dimension $m+1$. It can be represented by a cycle $\gamma_i \in \underline{\mathbf{L}}$ in degree 1. Using the Jacobi identity one can write

$$\gamma_i = \Sigma_{k=1}^{s} c_k u_k \text{ where } c_k \in R \text{ and } u_k = [\cdots[y_{j_k}, x_{k_1}], \ldots, x_{k_{n_k}}],$$

with $[x_{k_i}] \in L_X$.

The sphere S^m has an Adams-Hilton model $(\mathbb{T}\langle a \rangle, 0)$ which can be extended to $(\mathbb{T}\langle a, b \rangle, d)$ where $db = a$ which is an Adams-Hilton model for the disk D^{m+1}. Using properties of Adams-Hilton models, one can construct maps $g_k : (D^{m+1}, S^m) \to (Y, X)$ for $1 \leq k \leq s$ such that $\mathbf{A}(g_k)(b) = c_k u_k$.

Let

$$g' = g_1 \vee \ldots \vee g_s : \left(\bigvee_{k=1}^{s} D^{m+1}, \bigvee_{k=1}^{s} S^m \right) \to (Y, X).$$

Using the fact that $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$, one can show that $g'|_{\bigvee_k S^m}$ is contractible in X. As a result, g' can be extended to a map $g : S^{m+1} \to Y$ whose Hurewicz image modulo lower filtration is $\bar{w}_i$. $\square$

From this we will prove our final main result. Recall that $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ containing $\frac{1}{6}$.

Theorem 5.5. *Let $Y = X \cup_f \left(\bigvee e^{n_j+1}\right)$ with $f = \bigvee \alpha_j$ satisfying the hypotheses of Theorem 4.3. In addition assume that h_X has a right inverse σ_X and that the implicit primes are invertible. Then*
(a) the canonical algebra map

$$UL_Y \to H_*(\Omega Y; R) \tag{5.1}$$

is a surjection.
(b) If $R \subset \mathbb{Q}$ then (5.1) *is an isomorphism,* $\mathrm{gr}(L_Y) \cong L_Y^X \ltimes \mathbb{L}((H\underline{\mathbf{L}})_1)$*, and localized at R, $\Omega Y \in \prod \mathcal{S}$.*
(c) If $R \subset \mathbb{Q}$ and f is semi-inert *then*
(i) there exists $\hat{K} \subset F_1 L_Y$ such that $L_Y \cong L_Y^X \amalg \mathbb{L}\hat{K}$ as Lie algebras,
(ii) $L_Y \cong H\underline{\mathbf{L}}$ as Lie algebras, and
(iii) h_Y has a right inverse σ_Y.

Proof of (a) and (b).
(a) Let $g : \mathrm{gr}(H_*(\Omega Y; R)) \xrightarrow{\cong} UL'$ be the algebra isomorphism given by Theorem 4.3(a) where $L' = L_Y^X \ltimes \mathbb{L}K'$ with $K' = (H\underline{\mathbf{L}})_1$. Note that $L'_0 = L_Y^X$ and that $L'_1 = K'$.

We have an injection of Lie algebras

$$u_0 : L_Y^X \hookrightarrow F_0 L_Y \xrightarrow{\cong} (\mathrm{gr}(L_Y))_0. \tag{5.2}$$

Since the implicit primes are invertible, we have that for all j, $h_X \sigma_X \hat{\alpha}_j = \hat{\alpha}_j$. So by Proposition 5.4 we get an injection of L_Y^X-modules $u_1 : K' \hookrightarrow (\operatorname{gr}(L_Y))_1$. Hence for $x \in L_Y^X$ and $y \in K'$, $u_1([y,x]) = [u_1(y), u_0(x)]$. Thus u_0 and u_1 can be extended to a Lie algebra map $u : L' \to \operatorname{gr}(L_Y)$.

The inclusion $L_Y \hookrightarrow H_*(\Omega Y; R)$ induces a map between the corresponding graded objects, $\chi : \operatorname{gr}(L_Y) \to \operatorname{gr}(H_*(\Omega Y; R))$.

We claim that for $j = 0$ and 1, $g \circ \chi \circ u_j$ is the ordinary inclusion of L'_j in UL'. For $j = 0$, under the isomorphisms $\operatorname{gr}_0 L_Y \cong F_0 L_Y$ and $\operatorname{gr}_0 H_*(\Omega Y; R) \cong F_0 H_*(\Omega Y; R)$, $g\chi u_0$ corresponds to the map $L_Y^X \hookrightarrow F_0 L_Y \hookrightarrow F_0 H_*(\Omega Y; R) \xrightarrow{\cong} UL_Y^X$. For $j = 1$, under the isomorphism $UL' \cong HU\underline{\mathbf{L}}$, $g\chi u_1$ corresponds to the inclusion $K' = (H\underline{\mathbf{L}})_1 \hookrightarrow (HU\underline{\mathbf{L}})_1$. It follows that $g \circ \chi \circ u$ is the standard inclusion $L' \hookrightarrow UL'$. Since $g \circ \chi \circ u$ is an injection, so is u.

The canonical map $U \operatorname{gr}(L_Y) \xrightarrow{\cong} \operatorname{gr}(UL_Y)$ is an algebra isomorphism. Now u and χ induce the maps Uu and $\bar{\chi}$ in the following diagram.

$$
\begin{tikzcd}
UL' \arrow[r, "Uu"] \arrow[dr, "\cong", "g^{-1}"'] & U\operatorname{gr}(L_Y) \arrow[r, "\cong"] \arrow[d, "\bar{\chi}"] & \operatorname{gr}(UL_Y) \arrow[dl, dashed, "\tilde{\chi}"] \\
& \operatorname{gr}(H_*(\Omega Y; R)) &
\end{tikzcd}
\tag{5.3}
$$

Since we showed that $g\chi u$ is the ordinary inclusion $L' \hookrightarrow UL'$ the diagram commutes. Since g^{-1} is surjective, the induced map $\tilde{\chi}$ is surjective. Since $\tilde{\chi}$ is the associated graded map to the canonical map $UL_Y \to H_*(\Omega Y; R)$ and the filtrations are bicomplete, the associated ungraded map is also surjective. So the canonical map $UL_Y \to H_*(\Omega Y; R)$ is surjective which finishes the proof of (i).

(b) In the case where $R \subset \mathbb{Q}$ we can tensor with $\mathbb{Q}$ and make use of results from rational homotopy theory.

Recall that $\operatorname{gr}(H_*(\Omega Y; R)) \cong UL'$ and that we constructed a Lie algebra map $u : L' \to \operatorname{gr}(L_Y)$ and showed that it is an injection. We claim that for $R \subset \mathbb{Q}$, u is an isomorphism. $H_*(\Omega Y; R)$ and $\operatorname{gr}(H_*(\Omega Y; R))$ have the same Hilbert series. Also since $H_*(\Omega Y; R)$ is torsion-free, it has the same Hilbert series as $H_*(\Omega Y; \mathbb{Q})$. Let S be the image of $h_Y \otimes \mathbb{Q}$. Then S, L_Y and $\operatorname{gr}(L_Y)$ have the same Hilbert series. By the Milnor-Moore Theorem [MM65], $H_*(\Omega Y; \mathbb{Q}) \cong US$. So by the Poincaré-Birkhoff-Witt Theorem, S has the same Hilbert series as L', and hence $\operatorname{gr}(L_Y) \cong L'$ as R-modules. Since $u : L' \to \operatorname{gr}(L_Y)$ is an injection it follows that it is an isomorphism.

Using diagram (5.3) we get that (5.1) is an isomorphism.

The map u gives the desired Lie algebra isomorphism $\operatorname{gr}(L_Y) \cong L' = L_Y^X \ltimes \mathbb{L}(H\underline{\mathbf{L}})_1$.

By the Hilton-Serre-Baues Theorem [Bau81, Lemma V.3.10],[Ani89, Lemma 3.1], that (5.1) is an isomorphism is equivalent to the statement that localized at R, $\Omega Y \in \prod \mathcal{S}$. $\square$

Before we prove (c) we strengthen the result in (b) in the semi-inert case.

Lemma 5.6. *Let $R \subset \mathbb{Q}$. Assume Y is a space satisfying the hypotheses of Theorem 5.5. If furthermore f is semi-inert then there exists $\hat{K} \subset F_1 L_Y$ such that $L_Y \cong L_Y^X \amalg \mathbb{L}\hat{K}$ as Lie algebras.*

Proof. Assume that f is semi-inert. Recall the situation from Theorem 4.3(b). We have that $\mathrm{gr}(H_*(\Omega Y;R)) \cong UL'$ where $L' \cong L_Y^X \amalg \mathbb{L}K'$ where $K' \cong R\{\bar{w}_i\} \subset (\mathrm{gr}(H_*(\Omega Y;R)))_1$. For each $\bar{w}_i$ let $[w_i]$ be an inverse image under the quotient map $H_*(\Omega Y;R) \to \mathrm{gr}(H_*(\Omega Y;R))$. Let $K'' = \{[w_i]\}$. By Theorem 4.3(b) as algebras $H_*(\Omega Y;R) \cong UL''$ where $L'' = L_Y^X \amalg \mathbb{L}K''$ (see Theorem 3.12(b)). Since $K'' \xrightarrow{\cong} K'$, there is an induced Lie algebra isomorphism $L'' \xrightarrow{\cong} L'$. So $\mathrm{gr}(H_*(\Omega Y;R)) \cong H_*(\Omega Y;R)$ as algebras.

Recall that $K' = (H\underline{\mathbf{L}})_1$, so by Proposition 5.4 there exists $\hat{K} \subset F_1 L_Y$ such that $f : \hat{K} \xrightarrow{\cong} K'$, where f is the quotient map from (3.6). Let

$$\hat{L} = L_Y^X \amalg \mathbb{L}\hat{K}.$$

So $f : \hat{K} \xrightarrow{\cong} K'$, induces a Lie algebra isomorphism $\hat{L} \xrightarrow{\cong} L'$. This in turn induces the algebra isomorphism $U\hat{L} \xrightarrow{\cong} UL' \xrightarrow{\cong} H_*(\Omega Y;R)$.

Since $U\hat{L} \cong H_*(\Omega Y;R)$ as algebras there is an injection $L_Y \hookrightarrow U\hat{L}$. Also, since $\hat{K} \subset L_Y$ there is a canonical Lie algebra map $v : \hat{L} \to L_Y$. These fit into the following commutative diagram.

$$\begin{tikzcd} \hat{L} \arrow[rr] \arrow[dr, "v"'] & & U\hat{L} \\ & L_Y \arrow[ur] & \end{tikzcd}$$

It follows that v is an injection.

We claim that v is an isomorphism. Since $H_*(\Omega Y;R)$ is torsion-free, it has the same Hilbert series as $H_*(\Omega Y;\mathbb{Q})$. Let S be the image of $h_Y \otimes \mathbb{Q}$. Then S and L_Y have the same Hilbert series. By the Milnor-Moore Theorem [MM65], $H_*(\Omega Y;\mathbb{Q}) \cong US$. So by the Poincaré-Birkhoff-Witt Theorem, S has the same Hilbert series as $\hat{L}$, and hence $L_Y \cong \hat{L}$ as R-modules. Since $v : \hat{L} \to L_Y$ is an injection it follows that it is an isomorphism.

Therefore $L_Y \cong L_Y^X \amalg \mathbb{L}\hat{K}$ as Lie algebras, with $\hat{K} \subset F_1 L_Y$ and $H_*(\Omega Y;R) \cong UL_Y$ as algebras. □

Proof of Theorem 5.5(c). Lemma 5.6 proves (i) and puts us in a position to prove (ii) and (iii).

(ii) Recall that $(H\underline{\mathbf{L}})_1 \cong \hat{K}$ and $L' = (H\underline{\mathbf{L}})_0 \amalg \mathbb{L}((H\underline{\mathbf{L}})_1)$. Thus there is a canonical map $\nu : L' \to H\underline{\mathbf{L}}$. Now the composition

$$L' \xrightarrow{\nu} H\underline{\mathbf{L}} \xrightarrow{\phi} HU\underline{\mathbf{L}} \xrightarrow{\cong} UL'$$

is just the ordinary inclusion of L' into its universal enveloping algebra. Therefore ν is an injection. Tensoring with $\mathbb{Q}$ we get that L' and $H\underline{\mathbf{L}}$ have the same Hilbert series. It follows that ν is an isomorphism.

(iii) We will construct a map σ_Y right inverse to h_Y.

Let i denote the inclusion $X \hookrightarrow Y$. Consider the composite map

$$F : [L_X^W] \hookrightarrow L_X \xrightarrow{\sigma_X} \pi_*(\Omega X) \otimes R \xrightarrow{(\Omega i)_\#} \pi_*(\Omega Y) \otimes R.$$

We claim that $F = 0$. Since F is a Lie algebra map it is sufficient to show that it is zero on the Lie algebra generators of L_X^W. That is, show

$$(\Omega i)_\# \sigma_X(R\{h_X(\hat{\alpha}_j)\}) = 0.$$

Since there are no implicit primes $\sigma_X h_X \hat{\alpha}_j = \hat{\alpha}_j$. By the construction of Y, $\Omega i \circ \hat{\alpha}_j \simeq 0$. So $F = 0$ as claimed.

Therefore there is an induced map $G : L_Y^X \xrightarrow{\cong} L_X/[L_X^W] \to \pi_*(\Omega Y) \otimes R$. That $h_Y \circ G$ is the inclusion map can be seen from the following commutative diagram.

$$\begin{tikzcd}
L_X \arrow[d] \arrow[r, dashed, bend left, "\sigma_X"] & \pi_*(\Omega X) \otimes R \arrow[l, "h_X"] \arrow[dd, "(\Omega i)_\#"] \\
L_Y^X \arrow[d] \arrow[dr, "G"] & \\
L_Y & \pi_*(\Omega Y) \otimes R \arrow[l, "h_Y"']
\end{tikzcd}$$

Now construct $\sigma_Y : L_Y \to \pi_*(\Omega Y) \otimes R$ as follows. We have shown that $L_Y \cong L_Y^X \amalg \mathbb{L}\hat{K}$ for some $\hat{K} \subset F_1 L_Y$. Since $h_Y : \pi_*(\Omega Y) \otimes R \twoheadrightarrow L_Y$, choose preimages $\check{K} \subset \pi_*(\Omega Y) \otimes R$ such that $h_Y : \check{K} \xrightarrow{\cong} \hat{K}$. Let $\sigma_Y|_{L_Y^X} = G$ and let $\sigma_Y \hat{K} = \check{K}$ be right inverse to h_Y. Now extend σ_Y canonically to a Lie algebra map on L_Y.

We finally claim that $h_Y \sigma_Y = id_{L_Y}$. Since $h_Y \sigma_Y$ is a Lie algebra map it suffices to check that it is the identity for the generators.

$$h_Y \sigma_Y L_Y^X = h_Y G L_Y^X = L_Y^X, \quad h_Y \sigma_Y \hat{K} = h_Y \check{K} = \hat{K}$$

Therefore σ_Y is the desired Lie algebra map right inverse to h_Y. □

Remark 5.7. Anick conjectured Theorem 5.5(c)(ii) without the semi-inert condition in the special case where X is a wedge of spheres [Ani89, Conj. 4.9].

Corollary 5.8 (Corollary 1.6)**.** *If $R \subset \mathbb{Q}$ then the canonical algebra map*

$$U(F(\pi_*(\Omega Y) \otimes R)) \to H_*(\Omega Y; R)$$

is an isomorphism.

Proof. Recall that $FM = M/\operatorname{Torsion}(M)$. By definition, the Lie algebra map $h_Y : \pi_*(\Omega Y) \otimes R \to L_Y$ is a surjection. By Theorem 4.3, $L_Y \subset H_*(\Omega Y; R)$ is torsion-free. As a result, there is an induced surjection $\tilde{h}_Y : F(\pi_*(\Omega Y) \otimes R) \twoheadrightarrow L_Y$.

Furthermore if we tensor with $\mathbb{Q}$ we see that $\pi_*(\Omega Y) \otimes \mathbb{Q} \xrightarrow{\cong} L_Y \otimes \mathbb{Q}$, which is a result of Cartan and Serre (see [FHT01]). Thus $F(\pi_*(\Omega Y) \otimes R)$ and L_Y have the same Hilbert series, and therefore $\tilde{h}_Y$ is an isomorphism of Lie algebras.

So using Theorem 5.5(b), the canonical algebra map

$$U(F(\pi_*(\Omega Y) \otimes R)) \to UL_Y \xrightarrow{\cong} H_*(\Omega Y; R)$$

is an isomorphism. □

Corollaries 1.7 and 1.8 follow immediately from Theorem 5.5.

6. Examples

(Spherical) n-cones are those spaces X such that there exists a finite sequence

$$* = X_0 \subset X_1 \subset \cdots \subset X_n \simeq X$$

where for $k \geq 0$, X_{k+1} is the *adjunction space*

$$X_{k+1} = X_k \cup_{f_{k+1}} \big(\bigvee_{j\in J} e^{n_j+1}\big).$$

In particular, any finite CW-complex is an n-cone for some n.

Example 6.1. Let $X = S^3_a \vee S^3_b$ and let $\iota_a,\ \iota_b$ denote the inclusions of the spheres into X. Let $Y = X \cup_{\alpha_1\vee\alpha_2} (e^8 \vee e^8)$ where the attaching maps are given by the iterated Whitehead products $\alpha_1 = [[\iota_a, \iota_b], \iota_a]$ and $\alpha_2 = [[\iota_a, \iota_b], \iota_b]$.

Let $W = S^7 \vee S^7$ and for $i = 1, 2$ let $\hat{\alpha}_i : S^6 \to \Omega X$ denote the adjoint of α_i. Let $R = \mathbb{F}_p$ with $p > 3$ or $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Then $[L^W_X] = [R\{h_X(\hat{\alpha}_1), h_X(\hat{\alpha}_2)\}]$.

Y has an Adams-Hilton model (see Section 2) $U(L, d)$ where $L = \mathbb{L}\langle x, y, a, b\rangle,\ |x| = |y| = 2,\ dx = dy = 0,\ da = [[x, y], x]$ and $db = [[x, y], y]$. Furthermore $h_X(\hat{\alpha}_1) = [da]$ and $h_X(\hat{\alpha}_2) = [db]$.

It is well-known that over a field, a Lie subalgebra of a free Lie algebra is also free. Thus, since $\mathbb{L}\langle x, y\rangle$ is a free Lie algebra, UL is a free dga extension (in the sense of Definition 3.9).

Let $u = [a, y] - [b, x]$. Then $du = [[[x, y], x], y] - [[[x, y], y], x] = [[x, y], [x, y]] = 0$. Since u is not a boundary $0 \neq [u] \in (HL)_1$ and $0 \neq [u] \in (HUL)_1$. Thus UL is not an inert dga extension. By the definition of homology

$$(HL)_0 \cong \mathbb{L}\langle x, y\rangle \,/\, \big[R\{[[x, y], x], [[x, y], y]\}\big].$$

One can check that $(HL)_1$ is freely generated by the $(HL)_0$-action on $[u]$. Thus UL is a semi-inert extension. Therefore by Theorem 3.12(b),

$$HUL \cong U((HL)_0 \amalg \mathbb{L}\langle [u]\rangle)$$

as algebras. Thus as algebras

$$\begin{aligned} H_*(\Omega Y; R) &\cong U\left(L_X/[L^W_X] \amalg \mathbb{L}\langle [u]\rangle\right) \\ &\cong \left(H_*(\Omega X; R)/(L^W_X)\right) \amalg \mathbb{T}\langle [u]\rangle. \end{aligned}$$

$\alpha_1 \vee \alpha_2$ is a non-inert, semi-inert attaching map.

If $R = \mathbb{Z}[\frac{1}{6}]$ then since X is a wedge of spheres there exists a map σ_X right inverse to h_X. By Lemma 5.2, $P_Y = \{2, 3\}$. By Theorem 5.5,

$$H_*(\Omega Y; R) \cong UL_Y, \quad L_Y \cong L^X_Y \amalg \mathbb{L}\langle w\rangle \cong \mathbb{L}\langle x, y, w\rangle/J,$$

where $w = h_Y(\hat{\omega})$ with $\hat{\omega}$ the adjoint of some map $\omega : S^{10} \to Y$, and J is the Lie ideal generated by $\{[[x, y], x], [[x, y], y]\}$. Furthermore localized at R, $\Omega Y \in \prod \mathcal{S}$ and there exists a map σ_Y right inverse to h_Y. $\square$

Example 6.2. The 6-skeleton of $S^3 \times S^3 \times S^3$.

This space Y is also known as the *fat wedge* $FW(S^3, S^3, S^3)$. Let $X = S^3_a \vee S^3_b \vee S^3_c$. Let ι_a, ι_b and ι_c be the inclusions of the respective spheres. Let $W = \bigvee_{j=1}^3 S^5$. Then $Y = X \cup_f (\bigvee_{j=1}^3 e^6_j)$ where $f : W \to X$ is given by $\bigvee_{j=1}^3 \alpha_j$ with $\alpha_1 = [\iota_b, \iota_c]$, $\alpha_2 = [\iota_c, \iota_a]$ and $\alpha_3 = [\iota_a, \iota_b]$.

Let $R = \mathbb{Z}[\frac{1}{6}]$. Then Y has Adams-Hilton model $U(L, d)$ where $L = \mathbb{L}\langle x, y, z, a, b, c\rangle$, $|x| = |y| = |z| = 2,\ dx = dy = dz = 0,\ da = [y, z],\ db = [z, x]$ and $dc = [x, y]$. Let $w = [x, a] + [y, b] + [z, c]$.

By the same argument as in the previous example, as algebras

$$H_*(\Omega Y;R) \cong UL_Y, \quad L_Y \cong L_Y^X \amalg \mathbb{L}\langle w\rangle \cong \mathbb{L}\langle x,y,z,w\rangle/J,$$

where $w = h_Y(\hat{\omega})$ with $\hat{\omega}$ the adjoint of some map $\omega : S^8 \to Y$ and J is the Lie ideal generated by $\{[x,y],[y,z],[z,x]\}$. Furthermore localized at R, $\Omega Y \in \prod \mathcal{S}$ and there exists a map σ_Y right inverse to h_Y. □

The following *spherical three-cone* Y, illustrates our results.

Example 6.3. Let $R = \mathbb{Z}[\frac{1}{6}]$. All spaces here are localized at R. For $i = 1,2$ let $Z_i = A_i \cup_{\alpha_1 \vee \alpha_2} (e^8 \vee e^8)$ be two copies of the two-cone from Example 6.1. Let $X = Z_1 \vee Z_2$, $W = S^{28} \vee S^{28}$ and let $f = \beta_1 \vee \beta_2$ where $\beta_1 = [[\omega_1,\omega_2],\omega_1]$ and $\beta_2 = [[\omega_1,\omega_2],\omega_2]$. Let $Y = X \cup_f (e^{29} \vee e^{29})$.

Now,

$$L_X \cong L_{Z_1} \amalg L_{Z_2} \cong L_{Z_1}^{A_1} \amalg L_{Z_2}^{A_2} \amalg \mathbb{L}\langle w_1,w_2\rangle. \tag{6.1}$$

If follows from this that f is a free attaching map. Thus Y satisfies the hypotheses of Theorem 4.3. Using Theorem 4.3 and Anick's formula one can calculate that if f is semi-inert then $K'(z) = z^{37}$.

Recall that $\underline{\mathbf{L}} = (L_X \amalg \mathbb{L}\langle e,g\rangle, d')$ where $d'e = [[w_1,w_2],w_1]$ and $d'g = [[w_1,w_2],w_2]$. Also recall (from Theorem 4.3) that $(H\underline{\mathbf{L}})_0 \cong L_Y^X$. Let $\bar{u} = [e,w_2] + [g,w_1]$ (with $|\bar{u}| = 37$). Then one can check that $d\bar{u} = 0$ and $[\bar{u}]$ is a basis for $(H\underline{\mathbf{L}})_1$ as a free $(H\underline{\mathbf{L}})_0$-module; so f is indeed semi-inert. Let $\sigma_X = \sigma_{Z_1} \oplus \sigma_{Z_2}$. It is right inverse to h_X. By Lemma 5.2, $P_Y = \{2,3\}$. As a result by Theorem 5.5,

$$H_*(\Omega Y;R) \cong UL_Y \text{ where}$$
$$L_Y \cong L_Y^X \amalg \mathbb{L}\langle u\rangle \cong \mathbb{L}\langle x_1,y_1,x_2,y_2,w_1,w_2,u\rangle/J,$$

with $u = h_Y(\hat{\mu})$ for some map $\mu : S^{38} \to Y$ and J the Lie ideal generated by

$$\{[[x_1,y_1],x_1],\ [[x_1,y_1],y_1],\ [[x_2,y_2],x_2],$$
$$[[x_2,y_2],y_2],\ [[w_1,w_2],w_1],\ [[w_1,w_2],w_2]\}.$$

Furthermore $\Omega Y \in \prod \mathcal{S}$ and there exists a map σ_Y right inverse to h_Y.

Note that

$$L_Y \cong L_1^1 \amalg L_1^2 \amalg L_2 \amalg \mathbb{L}\langle u\rangle$$

where $L_1^1 \cong L_1^2 \cong R\{x,y,[x,y]\}$ and $L_2 \cong R\{w_1,w_2,[w_1,w_1],[w_1,w_2],[w_2,w_2]\}$. □

Example 6.4. An infinite family of finite CW-complexes constructed out of semi-inert attaching maps

The construction in the previous example can be extended inductively. By induction, we will construct spaces X_n and maps $\omega_n : S^{\lambda_n} \to X_n$ for $n \geq 1$ such that X_n is an n-cone constructed out of a sequence of semi-inert attaching maps. Given ω_n, let $w_n = h_{X_n}([\omega_n])$ and given w_i^a and w_i^b, let $L_i = \mathbb{L}\langle w_i^a, w_i^b\rangle/J_i$ where J_i is the Lie ideal of brackets of bracket length ≥ 3.

Let $R = \mathbb{Z}[\frac{1}{6}]$. Begin with $X_1 = S^3$ and $\lambda_1 = 3$. Let $\omega_1 : S^{\lambda_1} \to X_1$ be the identity map.

Given X_n, let X_n^a and X_n^b be two copies of X_n. For $n \geq 1$, let

$$X_{n+1} = X_n^a \vee X_n^b \cup_{f_{n+1}} (e^{\kappa_{n+1}} \vee e^{\kappa_{n+1}}),$$

where $\kappa_{n+1} = 3\lambda_n - 1$ and $f_{n+1} = [[\omega_n^a, \omega_n^b], \omega_n^a] \vee [[\omega_n^a, \omega_n^b], \omega_n^b]$.

By the same argument as in the previous example, f_{n+1} is a semi-inert cell attachment and there exists a map

$$\omega_{n+1} : S^{\lambda_{n+1}} \to X_{n+1}$$

where $\lambda_{n+1} = 4\lambda_n - 2$, such that

$$H_*(\Omega X_{n+1}; R) \cong UL_{X_{n+1}} \text{ where } L_{X_{n+1}} = \Big(\coprod_{\substack{1 \le i \le n \\ 1 \le j \le 2^{n-i}}} L_i^j \Big) \amalg \mathbb{L}\langle w_{n+1} \rangle$$

with L_i^j a copy of L_i and $w_{n+1} = h_{X_{n+1}}(\hat{\omega}_{n+1})$. □

Example 6.5. An uncountable family of CW-complexes constructed out of semi-inert cell attachments

At each stage of the inductive construction in Example 6.4, we could have used an attachment of the type in Example 6.2 instead of an attachment of the type in Example 6.1. For $\alpha \in [0, 1)$ use the binary expansion of α to choose the sequence of attachments to obtain a space X_α.

Example 6.6. CW-complexes X with only odd-dimensional cells

Let $X^{(n)}$ denote the n-skeleton of X. From the CW-structure of X, there is a sequence of cell attachments for $k \geq 1$.

$$W_{2k} \xrightarrow{f_{2k+1}} X^{(2k-1)} \to X^{(2k+1)}$$

where W_{2k} is a finite wedge of $(2k)$-dimensional spheres, $X^{(2k+1)}$ is the adjunction space of the cell attachment and $X^{(1)} = *$. Assume $R \subset \mathbb{Q}$ containing $\frac{1}{6}$. Let $P_{X^{(n)}}$ be the set of implicit primes of $X^{(n)}$ (see Definition 5.1). We will show by induction that

$$H_*(\Omega X^{(2k+1)}; \mathbb{Z}[{P_{X^{(2k+1)}}}^{-1}]) \cong UL_{X^{(2k+1)}}$$

where $L_{X^{(2k+1)}} \cong \mathbb{L}V^{(2k+1)}$ with $V^{(2k+1)}$ concentrated in even dimensions. In addition localized away from $P_{X^{(2k+1)}}$, $\Omega X^{(2k+1)} \in \prod \mathcal{S}$ and there exists a map $\sigma_{X^{(2k+1)}}$ right inverse to $h_{X^{(2k+1)}}$.

For $k = 0$ these conditions are trivial. Assume they hold for $k-1$. Let $\underline{\mathbf{L}} = (L_{X^{(2k-1)}} \amalg \mathbb{L}K, d')$ where K is a free R-module in dimension $2k$ corresponding to the spheres in W_{2k}. For degree reasons $L^{W_{2k}}_{X^{(2k-1)}} = d'(K) = 0$. So f_{2k+1} is automatically free. Furthermore $\underline{\mathbf{L}}$ has zero differential so $H\underline{\mathbf{L}} = \mathbb{L}V^{(2k-1)} \amalg \mathbb{L}K$. Thus f_{2k+1} is semi-inert. By Theorem 5.5, $H_*(\Omega X^{(2k+1)}; \mathbb{Z}[{P_{X^{(2k+1)}}}^{-1}]) \cong UL_{X^{(2k+1)}}$, where

$$L_{X^{(2k+1)}} \cong L^{X^{(2k-1)}}_{X^{(2k+1)}} \amalg \mathbb{L}K \cong L_{X^{(2k-1)}} \amalg \mathbb{L}K \cong \mathbb{L}(V^{(2k-1)} \oplus K).$$

Also by Theorem 5.5, localized away from $P_{X^{(2k+1)}}$, $\Omega X^{(2k+1)} \in \prod \mathcal{S}$, and there exists a map $\sigma_{X^{(2k+1)}}$ right inverse to $h_{X^{(2k+1)}}$.

Therefore by induction $H_*(\Omega X; \mathbb{Z}[{P_X}^{-1}]) \cong UL_X$, where $L_X \cong \mathbb{L}(s^{-1}\tilde{H}_*(X))$ with s the suspension map, and localized away from P_X, $\Omega X \in \prod \mathcal{S}$. □

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